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Sec 13

- Matrix algebra : If m, n are positive integers then an $m \times n$ matrix is a rectangular array

$$\underset{m \times n}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$m \times n$ = size of matrix

$m \to$ rows, $n \to$ columns

Special case : $m = n$ (square matrix)

$$\underset{3 \times 3}{A} = \begin{pmatrix} 3 & 2 & 5 \\ 1 & 0 & 3 \\ -2 & -3 & 4 \end{pmatrix}$$

The main diagonal

Equality of matrices : Two matrices $A(a_{ij})$ and $B(b_{ij})$ are equal if they have the same size $(m \times n)$ and $a_{ij} = b_{ij}$

Ex) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ 2×2 $\quad B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ 2×1 $\quad C = \begin{pmatrix} 1 & 3 \end{pmatrix}$ 1×2 $\quad D = \begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix}$ 2×2

- if $x = 4$ then A and D are equal, otherwise they are not equal
- $B \neq C$ because they don't have same size

<u>Matrix of one row or one column</u>

$A = \begin{pmatrix} 3 & 2 & 5 & 8 & 0 \end{pmatrix}$ row matrix (row vector)

$B = \begin{pmatrix} 8 \\ 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}$ Column matrix (column vector)

Matrix operations

1- Matrix addition

Two matrices are added only if they have same size

ex

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 3 \end{pmatrix}$$

$$\underset{2\times 2}{A} + \underset{2\times 2}{B} = \begin{pmatrix} 0 & 5 \\ -1 & 3 \end{pmatrix} \qquad \underset{2\times 2}{A} + \underset{1\times 2}{C} = \text{undefined}$$

2- Scalar multiplication

When multiplying a matrix with a number, we multiply all entries in that matrix

ex) $A = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix} \qquad 3A = \begin{pmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 9 \end{pmatrix}$

3- Matrix multiplication

$$A = \begin{pmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \qquad \underset{3\times 2}{A} \times \underset{2\times 2}{B} = \underset{3\times 2}{C}$$

equal

For two matrices to be multiplied, the no. of columns in first matrix is the same as no. of rows in the second matrix

A × B can be multiplied $\quad 3\times 2 \times 2\times 2$

B × A can't be multiplied $\quad 2\times 2 \times 3\times 2$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{pmatrix}$$

To get C_{11} multiply row 1 in A by column 1 in B

To get C_{31} multiply row 3 in A by column 1 in B

to get C_{22} multiply row 2 in A by column 2 in B

* The product matrix has no. of rows of first matrix and no. of columns of the second matrix

$$A = \begin{pmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{pmatrix} \times B = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} = C = \begin{pmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{pmatrix}$$

$C_{11} = (-1 \times -3) + (3 \times 4) = 9$ $\qquad$ $C_{21} = (4 \times -3) + (-2 \times -4) = -4$

$C_{12} = (-1 \times 2) + (3 \times 1) = 1$ $\qquad$ $C_{22} = (4 \times 2) + (-2 \times 1) = 6$

Ex. $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ $\qquad$ $B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$

$$\underset{2\times 2}{A} \times \underset{2\times 2}{B} = \underset{2\times 2}{C} \begin{pmatrix} 2 & 5 \\ 4 & -4 \end{pmatrix}$$

Ex.

$$\underset{3\times 3}{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 5 & 0 & 2 \end{pmatrix}} \cdot \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = \underset{3\times 1}{\begin{pmatrix} 4 \\ 10 \\ 11 \end{pmatrix}}$$

Ex

$$\underset{2\times 4}{\begin{pmatrix} 1 & 2 & 3 & 0 \\ -1 & 0 & 1 & 3 \end{pmatrix}} \cdot \underset{4\times 2}{\begin{pmatrix} 5 & 1 \\ 0 & 2 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 2 & 5 \\ -3 & 2 \end{pmatrix}}$$

properties of matrix operations

1- $A + B = B + A$

2- $AB \neq BA$ in general

3- $A + (B + C) = (A + B) + C$

4- $c(A + B) = cA + cB$

5- $A(BC) = (AB)C \neq (AC)B$

6- $A(B + C) = AB + AC$

Note: $(A+B)(A-B) \neq A^2 - B^2$

Ex. $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$ Find AB, BA

$AB = \begin{pmatrix} 2 & 5 \\ 4 & -4 \end{pmatrix}$ $BA = \begin{pmatrix} 0 & 7 \\ 4 & -2 \end{pmatrix}$ $AB \neq BA$

Ex $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 4 \\ 2 & -2 \end{pmatrix}$ find AB, BA

$AB = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $BA = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $AB = BA$
~~special~~ rare case

If $AC = BC$, ~~$A \neq B$~~ then A not necessary equals B

Ex $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ $C = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$

$AC = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$ $BC = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$

$AC = BC$ but $A \neq B$

<u>Identity matrix</u>

The identity matrix of order (n) is a square matrix that has 1's on the main diagonal and 0's elsewhere

ex $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ I_2

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ I_3

$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

* When we multiply any matrix by identity matrix, the matrix does not change

eg. $\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$

The transpose of a matrix

$A = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 0 & -3 & -2 & 4 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 2 & 0 \\ 1 & -3 \\ 3 & -2 \\ 5 & 4 \end{pmatrix}$

$\cancel{4}\,2 \times 4 \qquad 4 \times 2$

Properties

1- $(A^T)^T = A$

2- $(A+B)^T = A^T + B^T$

3- $(AB)^T = B^T A^T$

4- if $A = A^T \Rightarrow$ Symmetric matrix

if $A = -A^T \Rightarrow$ Skew-Symmetric matrix